THE RELATION OF LOGIC TO MATHEMATICS.

A survey of the growth of the kindred sciences, logic and mathematics, leads inevitably to the conclusion that the two are inseparably connected. Exact thought seems to contain an inherent dialectic, of such character that mathematics without logic is formless, —and logic without mathematics is empty. The historical beginning of mathematics in the days of Ahmes, the pebbles in the sand, and the abacus, shows that it arose essentially from practical needs; its problems and symbols in this primitive stage must be regarded as essentially relevant to physical acts and objects. Arithmetic and geometry on this level of culture, form necessary means of adjustment to concrete phenomena and are as independent of human volition as any thing could well be. However, we find conflicts constantly cropping up; the budding science is unhandy and cumbersome in its very lack of coherency, and from practical reasons, we find a demand that mathematics be reduced to a coordinated whole.

As a result order and system gradually replace the heterogeneous complex of practical problems, and we come to the development of the Euclidean system, based upon axioms which are justified by their self-evidence. The close intimate connection with the physical world-order fades into obscurity; the lines and figures of geometry are not bodies or parts of bodies but independent mental constructions. Finally the Kantian criticism expounds the view that the axioms are true, not because they apply to the external and independent world-order, but because they are *necessarily* bound up with our forms of thought, because we can think in no other way without contradiction. Whereas at first human choice had nothing to do with mathematics because of the hard, unyielding character of physical fact, under the critical view, it is summarily dismissed because of our total inability to alter the make-up of our understanding.

The discovery of non-Euclidean geometry has gradually created a revolution in this point of view. It is seen with ever-increasing clearness that a decision of the will is involved in the establishment of each set of axioms, and that choices are involved which no future experiment can condemn or justify. For example the amount by which the angle-sum of a triangle differs from two right angles

may be so small as to baffle any observation, and it is thus entirely possible that we are dwellers of a non-Euclidean world. From a study of the objective and independent, mathematics thus becomes more and more concerned with a subjective element of arbitrary choice. Self-evidence disappears as a criterion, and is given its death-blow by Weierstrass in his brilliant example of a continuous curve which does not have a tangent.

Freed in this way, a fact which is at once a privilege and a restraint, mathematics develops in a double direction toward rigor and formalism. The growth of rigor may seem opposed to the increase of choice and independence of physical fact, but in reality the two are closely connected. Just as discipline is more easily effected in an organized army than among a hundred disconnected bands of outlaws, so, as mathematics becomes gradually reduced to a single rigorous system, it becomes increasingly submissive to human will, and develops a plasticity seemingly far removed from the condition of the hard, unyielding physical fact. From the time of Weierstrass, the tendency has been very strong to unify mathematics in terms of dependence upon the natural number system of positive integers. Any set of axioms defining a special part of the science is regarded as dependent in some way upon this natural number system in regard to consistency, categoricity and mutual independence. In this way the remote parts and latest developments of mathematics are elevated to the level of the elementary positive integers, gaining in certainty with the progress of rigor.

Combined with this, we have the counter-movement toward formalism, which brings with it the growing realization of the importance of the subjective element of choice. The principle of the so-called "permanence of forms," which accounts in a rather romantic way for the introduction of irrational, ideal, and imaginary numbers by attributing it to a demand which the mathematician makes upon experience, the study of groups as they appear in the different fields of order, the emphasis on the arbitrary and relative character of the indefinables,—all these lead to confidence in the power of mathematical creation and the adoption of the point of view that the positive integers are only a particular illustration of a peculiar kind of mathematical type of order, of which there are many other representations. Here we have commendable harmony and synthesis in mathematics, but it is not gained without cost. We must remember that if the remote parts of mathematics are just

as certain as the system of positive integers, by the very identical reasoning it follows that these same positive integers are no more certain than the remote parts of mathematics.

More than this, just as each part of mathematics depends upon its own postulates and indefinables, so does the natural number system in its turn. The doctrine of formalism tells us that a term is indefinable and a proposition postulated only in reference to a particular system of axioms and terms, in another system we have the terms defined and the propositions proved. Because of this relativity we may well inquire as to the essential nature of this number system upon which mathematics is supposed to depend. Is it itself something plastic, responsive to our demands, or is it objective and independent? The formalist is forced to admit that an infinity of different sets of axioms for the positive integers are possible. Why should we use any particular one of these rather than another? If no single one is to be preferred, and the essence of arithmetic lies in the theorems, and not in the postulates and indefinables, then our choice seems to have nothing to do with the matter. Confronted with this situation, the formalist is forced to an extreme position and enunciates the theory of *nominalism*,—that numbers, and, in fact, mathematical entities in general, are nothing more than mere words; that they are not real as physical objects are real, but are free creations of the spirit with no necessary connection with objective existence. Not use, interpretation, or application is the goal of mathematics, but consistency, alone; and apart from the requirements of consistency, all limitations are to be cast aside as fetters on the intellect.

This radical nominalism is, it should be noted, in distinct and admitted opposition to the original character of the science as it developed from the study of the hard and fast physical problem. The only law which is now recognized is the law of logical consistency, and the mathematician believes himself to have attained his goal of independence and freedom. Yet wherein can he boast of freedom? Divorced from restraint and reality, he seems to be reduced to contemplation and admiration of his own mental processes, and submissive to self-imposed rules alone. As Brunschvig aptly remarks, "Mathematics has lost its claim to be a science, for science at least pretends to truth."

The temporary success of this impulse toward freedom is not without corresponding reaction. We see Frege indignant against

the possibility of the subjective and arguing that the mathematician creates just as little as the geographer, that he can only describe what is *there*; Royce comparing the search of the arithmeticians for primes to the labors of the astronomers and impressed with the obstinate character of mathematical material. Indeed self-imposed fetters are often the most unyielding of bonds; witness the extremities to which a man will go for the sake of his honor, to which no external necessity could drive him.

Frege's attack upon this distasteful situation is at its most vulnerable point. If the mathematician is submissive to logic alone, that does not mean that the non-logical part of mathematics is subject to individual caprice, but that it does not exist. *Mathematics is objective and non-plastic because it is a part of logic.* Because the axioms of arithmetic can be proved from those of logic and the indefinables defined in logical terms, arithmetic and the whole of analysis has the certainty and consistency of logic itself. In this haven of rest Frege and Russell at first feel themselves completely satisfied.

But not for long. The inherent march of the dialectic is not to be halted. The supposedly safe ground of logic turns out to be a quicksand of contradictions, for the very reduction of mathematics to logical form has necessitated the employment of the notion of the class in such a way that from this and kindred notions there arise a veritable plague of paradoxes and contradictions. Mathematics may have all the certainty of logic, but these specimens of logic seem to be infected with an uncertainty far worse than any one had ever suspected to be present in the exact sciences.

Russell's remedy is swift and drastic. Logic is to be so changed that the paradoxes are avoided. The indefinables and primitive propositions, consequently, may be destitute of self-evidence (the axiom of reducibility, for example, of which one can only say at best that it has "every chance of being true"),—but that is a triviality compared with the victory over the paradoxes. However, when we find in the *Principia*: "Some propositions must be assumed without proof, since all inference proceeds from propositions previously asserted.... These, like the primitive ideas, are to some extent a matter of arbitrary choice," we may be pardoned for concluding that the work has not attained its goal, and that the hoped-for objectivity has not been reached. In fact, instead of an over-individual independent logic we have the logic of Russell and the

logic of Frege and the logic of McColl and the "new logic" of Mercier and the anti-logic of Schiller, so that in making logic the master we allow caprice to play a more important part than ever even "mathematical creation" played in mathematics.

But this is impossible, it may be contended. Russell and Frege may err in details but logic itself is serene and independent. It is the articulate expression of *the way men ought to reason.* Just so long as this contention remains in its pristine vagueness, is the adherent of such a theory safe from attack;—but let him descend from his abstractions and formulate a particular code, that *thus* and *so* is the way men ought to reason, and he will be objected to from all sides. The proponent of the code is unable to justify his concrete propositions except by reiteration of their self-evidence and his own personal convictions; he is prevented from appealing to the verdict of experience by very definition, since in no way can we derive what ought to be from what is, or has been. That "the way men ought to reason" gives logic an *ideal* is not to be seriously objected to, but this tells us no more than to be logical is to be logical.

From this we may turn to the opposite extreme, from the empty abstraction to the concrete event; from such a view-point logic is independent of human volition because it is the formal expression of the way men *do* reason. But this is a theory which is even less tenable than the preceding, for no one can deny that men reason illogically. That whatever is concluded, is logically concluded is no more to be accepted than the equally radical ethical doctrine, that whatever is, is right.

We are confronted with a situation that is far from promising. Mathematics is reduced to arithmetic and arithmetic in great part at least, to logic; but logic is so far from being independent of the will, as the mathematician would have it, that it seems to be neither the way men ought to reason nor the way men do reason, but something as yet entirely indeterminate. The mathematician is in an intolerable situation. He has gained freedom, it is true, but a freedom which is worse than the dependence of the physical scientist upon observation and experiment,—because it is not a freedom which has been wrested from a grudging opponent, but a freedom which he has bestowed upon himself. And in such an achievement there is little glory.

Nevertheless the chess-player and the mathematician must be

placed in different categories, for the true mathematician is not willing to be considered the devotee of "playing the game." He feels the *significance* of concrete experience. If he invents a branch of mathematics and cannot show either an interpretation for his undefined symbols or an application in the treatment of some pre-existing problem, his work will be liable to neglect and rejection. It is demanded of him that he *demonstrate* the consistence of his axioms by concrete illustrations, and he in his turn demands of logic that it give him a definition of consistency. Once this definition is gained, he considers himself bound by it. The logic, however, must in its turn be justified. How can its postulates be shown to be consistent and mutually independent? The answer is short and to the point, they *cannot*.

We are led to a new point of view. Logic does not contain an "ought," nor yet an "is"; it is an *accepted code of validity,* a kind of gentlemen's agreement, the violation of which should lead to scientific ostracism. So much in the abstract, but how are we to give this code content? If individual choice is allowed to interfere, we destroy the acceptability of the code, its universality. The Gordian Knot is not to be loosened; it must be cut. Just as the proof of the pudding is in the eating, so the justification of logic is in its use; we guarantee the acceptability of the code by defining it as what is accepted.

Thus if we are to give logic a progressively definite content we must condemn it to a patient abstraction from, and analysis of, the best usage of science. The logician, from this point of view, no longer dwells in lofty *a priori* solitude, holding absolute dominion over science and mathematics, but descends to the uncomplaining study and interpretation of the content of these subjects, having learned that true mastery is to be gained by service. Thus we are led again to the sphere of the concrete event and we might be supposed to have returned to our starting-place. The process is not a circle, however, and this new point of view is infinitely higher than the first. When we fully realize that consistency itself is given content by the *will to be consistent,* we see how it is that mathematics is an expression of that will exercising itself upon and in an independent world. The freedom which is present is not a freedom which is to be desired, but which is to be resolutely denied. We do not find the code of Aristotle springing full-born from the depths of his subjective experience; it is the result of his and Plato's study

of the biology and mathematics of their day, and no one would term the pre-Aristotelian geometry illogical because when it was invented, there was no such thing as logic.

Logic is the articulate development of the determination to be rational, and the progress of exact thought brings with it an increasing comprehension of the fact that the will to be rational is at once its motive power and its goal. The aim toward independence of the subjective does not give us independence, but it keeps us moving in that direction in which independence lies,—for he alone is a slave who is content to be one. The study of scientific and mathematical method may not give us forthwith our goal of a universally accepted code; we may not be able to realize the conception of a time when the way men ought to reason, the way men do reason, and the accepted code will coincide in part and whole. Yet it is no small thing that we are able to appreciate that the march of rational thought lies along this line, that logic is given content by the impulse toward rationality and that he knowledge of this content is to be gained by a study of the process itself. We are at least in possession of the differential coefficient.

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INDEPENDENCE PROOFS AND THE THEORY OF IMPLICATION.

Mathematical or Symbolic Logic claims to present a theory of deduction. It is the thesis of this article that the traditional symbolic logic, represented by the system of Whitehead and Russell's *Principia Mathematica,* fails to give a correct theory of mathematical deduction. It will be shown that the notion of mathematical deduction involved in the recognized methods of proving the independence of sets of postulates is incompatible with the theory of deduction furnished by mathematical logic.

Modern workers on the foundations of mathematics have formulated certain methods for the proof of the independence of a set of assumptions or postulates for a deductive theory. In a set of independent postulates no one of the postulates is deducible from the others. To prove the independence of a set of assumptions one must show, for every assumption of the set, that it cannot be derived as a formal consequence from the other assumptions. Hunting-